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High energy evolution of soft gluon cascades

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Abstract. In this paper we derive an evolution equation for the gluon density in soft gluon cascades emitted from any colored source, in the leading logarithmic approximation of perturbative QCD. We show that this equation has the same form as the BFKL equation in the forward case. An explicit expression for the total cascade wavefunction involving an arbitrary number of soft gluons is obtained. Renormalization of the colored source wavefunction turns out to be responsible for the reggeization of the source.

1 Introduction

The investigation of the hard scattering amplitude in the kinematics where the invariant energy s is large while the transfered momentum t is fixed is an important problem of QCD. The solution of this problem in the leading logarithmic approximation (LLA), collecting terms of the form $(\alpha_S \ln s)^n$, was proposed about twenty five years ago, in the BFKL equation [1]. These original papers were based on an effective triple gluons vertex and a bootstrap condition in the t-channel. This approach showed the relation between perturbative QCD and the reggeon calculus, which was proposed a decade before. The main feature of this work is the reggeization of the gluon, which appears not to be elementary but composite, being a pole in the complex momentum plane, with a color octet quantum number. Another important result is that the pomeron occurs as a bound state of two reggeized gluons in the singlet color channel. Although the BFKL equation looks like a ladder-type equation, it effectively sums up an infinite number of t-channel gluons.

A few years ago a dipole picture was proposed [2–7]. It is based on subsequent emissions of soft gluons, which iterates in the large N_c limit. In this approach elementary degrees of freedom are made of the colored dipole. The equation for the dipole density in this soft cascade turns out to be identical to the BFKL equation, although it was found in the multicolor limit, while the original BFKL was derived for finite N_c.

In this paper we study distributions of soft gluons produced by a fast moving colored source. Our treatment is based on the classical soft emission vertex which also underlies the dipole model [8]. However, we deal with gluons at finite N_c rather than with dipoles at large N_c, and we do not rely on the dipole notion in our approach. We show that these distributions satisfy an evolution equation with respect to the IR cut-off. This equation is similar to the DGLAP equation [9–12] except that the evolution variable is the longitudinal IR cut-off rather than the transverse cut-off. The obtained equation for the density of soft gluons has the same form as the forward BFKL equation. In the case of scattering, a similar evolution equation with respect to the longitudinal scale was obtained in [13].

Our paper is arranged as follows. After preliminary recalling the linear classical cascades properties, we generalize them in order to incorporate non-abelian effects in the LLA approximation. The main idea is that, despite the complicated internal structure of the cascade of soft gluons, the amplitude of soft emission from any gluon- (quark-) like source is the same as the one off a single gluon (respectively quark). This notion of soft classical cascades emitted by eikonal sources has been elaborated before by various authors [14–16]. Since we will intensively use this notion, we give details of this idea for consistency. Using this idea we obtain an equation describing the gluon density in a colored source. The same technique, when applied to QED, gives the same type of equation for the photon density inside a charged object. We then give an explicit expression for the total cascade wavefunction involving arbitrary numbers of soft gluons, similar to the multi-reggeon formula [1]. As an illustration, using this expression we reproduce the double logarithmic Sudakov form factor.

Our main results concern the relationship between particle reggeization in abelian and non-abelian theories and their self-energy, and the fact that the cascade wavefunction has a structure similar to the multi-reggeon formula.

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2 Classical cascade

The asymptotic behavior of the scattering cross sections are determined by the parton cascade from incoming particles or, in other words, by their wavefunction on the light cone. For a large invariant energy s (the so-called Regge limit), the amplitude is dominated by t-channel gluons rather than quarks, because of their largest spin. As a consequence gluons dominate in every channel, and quark contributions will be neglected below.

We shall consider a semi-hard region where gluons can be treated as soft with respect to the large invariant energy s, although still perturbative. It is known that in any massless field theory scattering amplitudes are dominated by IR contributions of two types, namely collinear and soft singularities. This semi-hard region reveals soft gluons contributions, which sum up into LLA. The physics behind soft resummation is that the field of an ultrarelativistic source can be treated as a state of almost real particles while the vertex for real or virtual soft gluon emission can be taken to be the same.

Let us denote by p_A and p_B the momenta of the colliding particles A and B. Neglecting their masses in the high energy limit we can take $p_A^2 = p_B^2 = 0$ so that the invariant energy reads $s = 2p_A \cdot p_B$. The vectors p_A, p_B can be used for the Sudakov decomposition of any vector as follows: $k = \alpha p_A + \beta p_B + k_\perp$.

Each particle develops a parton cascade, so the scattering of the incoming particles A and B could in principle be reduced to elementary parton processes. In this paper, we will focus on the structure of these cascades rather than on the scattering itself. The complete description of the cascade is a very complicated problem involving all perturbative as well as non-perturbative effects. There are certain limits where the cascade looks more simple. A well known example is deep inelastic scattering. For large virtuality Q^2 the parton density obeys the DGLAP evolution equation collecting the powers of $\alpha_S \ln Q^2$. Although predictions based on the DGLAP evolution yield good agreement with the present experimental data, this approach fails for parametrically small Bjorken variables x, when the powers of $\alpha_S \ln 1/x$ dominate. This is the region of Regge kinematics where the total invariant energy is much greater than the other invariants, including the virtuality of the deeply virtual photon. The leading $\ln 1/x$ behavior of the hard scattering amplitude is given in this region by the BFKL theory.

An important feature of the Regge kinematics is that the partons are soft there in the sense that the main contribution appears for small longitudinal variables, $\alpha, \beta \ll 1$. Only soft emissions from the external incoming or outgoing particles should be taken into account in this approximation. Since the momenta of all incident external particles are supposed to be along the p_A or p_B directions, two peculiar gauges are of special interest, namely the gauges $p_A \cdot A = 0$ and $p_B \cdot A = 0$. These gauges suppress soft emission from p_A and p_B lines respectively. In the following we shall deal with emission from a p_B line in the gauge $p_A \cdot A = 0$.

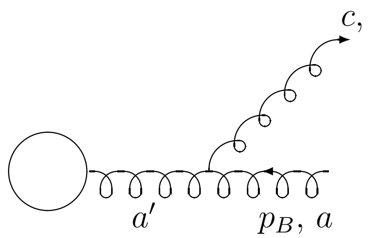

Fig. 1. Soft vertex for incoming line. The blob symbolizes any amplitude connected to the incoming line

The vertex for emitting a soft particle is rather universal and determined by the classical current proportional to the momentum of the source. It can be easily obtained, for example, from the triple gluon vertex in the soft limit. Consider emission off the incoming particle B, as illustrated in Fig. 1.

This incoming line is attached to an amplitude (shown as a blob in Fig. 1) whose peculiar form plays no role so long as we are dealing with soft emission. The soft vertex reads

$$\Gamma_c^\lambda(k) = g\frac{p_B \cdot \varepsilon^\lambda(k)}{p_B k - \mathrm{i}\delta} T_c = g\frac{p_B \cdot \varepsilon^\lambda(k)}{\alpha\frac{s}{2} - \mathrm{i}\delta} T_c, \tag{1}$$

where $\varepsilon^\lambda(k)$ is the helicity vector of the soft emitted or absorbed gluon carrying momentum k and color index c. The matrix T^c depends on the representation of the color group. For a gluon-like object it is expressed through the structure constants, $T^c_{a'a} = \mathrm{i} f_{a'ca}$, the indices c, a', a being in the adjoint representation.

The amplitude for the emission or absorption of n gluons (see Fig. 2) is given by the product of the elementary vertices,

$$\begin{aligned} &\Gamma^{\mu_n,\dots\mu_1}_{c_n,\dots c_1}(k_1,\dots k_n) A_{\mu_n,c_n}(k_n)\cdots A_{\mu_1,c_1}(k_1) \qquad (2)\\ &= (T^{c_n}\cdots T^{c_1})_{a'a}\, g\frac{p_B\cdot A_{c_n}(k_n)}{(\alpha_1+\alpha_2+\ldots+\alpha_n)\frac{s}{2}-\mathrm{i}\delta}\cdots \\ &\quad\times g\frac{p_B\cdot A_{c_2}(k_2)}{(\alpha_1+\alpha_2)\frac{s}{2}-\mathrm{i}\delta}\, g\frac{p_B\cdot A_{c_1}(k_1)}{\alpha_1\frac{s}{2}-\mathrm{i}\delta}. \end{aligned}$$

Here the field A is the asymptotic free field

$$\begin{aligned} A_{\mu,c}(x) &= \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{2k_0} \\ &\times\left[\mathrm{e}^{\mathrm{i}kx}\varepsilon^\lambda_\mu(k) a^+_{\lambda,c}(k) + \mathrm{e}^{-\mathrm{i}kx}\varepsilon^\lambda_\mu(k) a_{\lambda,c}(k)\right], \end{aligned}$$

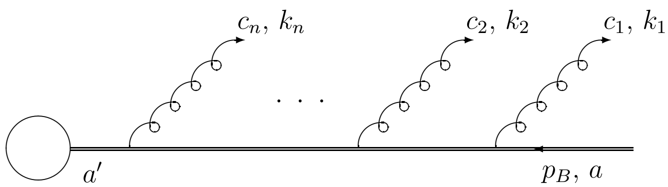

Fig. 2. Amplitude for n gluon emission or absorption off an incoming gluon-like object, represented as two thin lines

whose positive and negative frequency part corresponds respectively to the emission and absorption of gluons. Using the relation

$$\frac{1}{\alpha_1 \frac{s}{2} - \mathrm{i}\delta} = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}z \, \mathrm{e}^{\mathrm{i}\alpha \frac{s}{2} z} \theta(-z),$$

we get

$$\begin{aligned} &\int \mathrm{d}^4 k_1 \cdots \mathrm{d}^4 k_n \\ &\quad \times \Gamma^{\mu_n,\ldots\mu_1}_{c_n,\ldots c_1}(k_1,\ldots k_n) A_{\mu_n,c_n}(k_n) \cdots A_{\mu_1,c_1}(k_1) \\ &= \frac{(\mathrm{i}g)^n}{n!} \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ &\quad \times P\big[n_B \cdot A(z_n n_B) \cdots n_B \cdot A(z_1 n_B)\big]_{a'a}, \end{aligned}$$

with

$$A_\mu(x) = \int \mathrm{d}^4 k \, \mathrm{e}^{\mathrm{i}kx} A_{\mu,c}(k) T^c$$

and where the symbol P means the conventional ordering of the fields. The sum over the arbitrary number of soft gluons is given by a P-exponent along the momentum direction of the parent particle:

$$\begin{aligned} &\sum_n \int \mathrm{d}^4 k_1 \cdots \mathrm{d}^4 k_n \\ &\quad \times \Gamma^{\mu_n,\ldots\mu_1}_{c_n,\ldots c_1}(k_1,\ldots k_n) A_{\mu_n,c_n}(k_n) \cdots A_{\mu_1,c_1}(k_1) \\ &= P \exp\left[\mathrm{i}g \int_{-\infty}^{0} \mathrm{d}z \, n_B \cdot A(z n_B)\right]. \end{aligned} \tag{3}$$

A similar computation for an outgoing line results in replacing in the previous formula $\int_{-\infty}^{0}$ with $\int_{0}^{\infty}$. These formulae are nothing more than the Wilson line taken on the trajectory along the momentum of the incoming (respectively outgoing) particle. The scattering of heavy quarks can be described through the interaction of Wilson lines [17,18]. In the large s limit, following this approach one can make contact with BFKL physics. In particular the reggeon trajectory has been calculated up to two loop order [17].

3 Inclusion of interaction among emitted particles

The previous computation only takes into account emissions from the parent line which is the incoming line (respectively outgoing). The above formulae are strictly speaking only valid for QED. They do not take into account the interaction between emitted particles. In QCD, in the soft approximation, these corrections correspond to subsequent decays of the emitted gluons. Indeed in this approximation only end line emissions contribute, and thus to get the total cascade tree-level amplitude each field in the P-exponent has to be replaced by the P-exponent itself. Generally this turns into a complicated non-linear equation. Another problem is to incorporate loop corrections. This second problem will be considered in Sect. 4.

The first problem, the partons' subsequent decays, simplifies in the Regge kinematics implying the longitudinal momenta to be small in the formula (2), $\alpha, \beta \ll 1$, while

$$\alpha s, \beta s \gg k_\perp^2, \quad \alpha\beta s \sim k_\perp^2, \tag{4}$$

where $k_\perp$ is a typical transverse momentum scale. This kinematics ensures the soft vertex to be of the form (1) regardless from where a particular gluon is emitted off or where it is absorbed in. Indeed, the emission of a gluon with momentum $k_2 = \alpha p_A + \beta p_B + k_{2\perp}$ off the parent gluon carrying momentum $k_1 = a p_A + b p_B + k_{1\perp}$ is again given by a soft vertex similar to (1):

$$\Gamma^\lambda_c(k_2) = g \frac{k_1 \cdot \varepsilon^\lambda(k_2)}{k_1 \cdot k_2} T_c. \tag{5}$$

In the light cone gauge $p_A \cdot A(x) = 0$, the polarization vector reads

$$\varepsilon^\lambda_\mu(k) = \varepsilon^\lambda_A(k) \cdot p_{A\mu} + \varepsilon^\lambda_{\perp\mu}(k), \tag{6}$$

with

$$\sum_\mu \varepsilon^\lambda_{\perp\mu}(k) \cdot \varepsilon^{\lambda'}_{\perp\mu}(k) = \delta^{\lambda\lambda'}.$$

The emitted gluons are on-shell. Anyway, the gluons which dominate the amplitude in the Regge limit are soft and thus quasi-real; therefore the on-shell condition is unessential. The transversality condition for these quasi-real gluons reads $k \cdot \varepsilon = 0$, which implies for the helicity vector (6)

$$\varepsilon^\lambda_A(k) = 2 \frac{k_\perp \cdot \varepsilon^\lambda_\perp(k)}{\beta s}. \tag{7}$$

In this soft gluon approximation, $\beta \ll b$ and assuming the typical transverse momenta to be of the same order, $k_{1\perp}^2 \sim k_{2\perp}^2$, the numerator of the soft vertex is approximated as

$$k_1 \cdot \varepsilon^\lambda(k_2) = b \varepsilon^\lambda_A \frac{s}{2} - k_{1\perp} \varepsilon^\lambda_\perp \approx b p_B \cdot \varepsilon^\lambda(k_2).$$

Using the mass-shell conditions, $a = k_{1\perp}^2 / bs$, $\alpha = k_{2\perp}^2 / \beta s$, the denominator also simplifies as

$$k_1 \cdot k_2 = (a\beta + b\alpha) \frac{s}{2} - k_{1\perp} \cdot k_{2\perp} \approx b\alpha \frac{s}{2}.$$

It follows that the vertex (5) can be written as

$$\Gamma^\lambda_c(k_2) = g \frac{p_B \cdot \varepsilon^\lambda(k_2)}{p_B \cdot k_2} T_c, \tag{8}$$

which has exactly the universal form (1). Here, the vector p_B is the momentum of the source. The result (8) confirms the physical picture that soft emission is determined by the total current of the source rather than by its internal structure. Since all $k_\perp$ are assumed to be of the same order, the strong β ordering considered here is a special

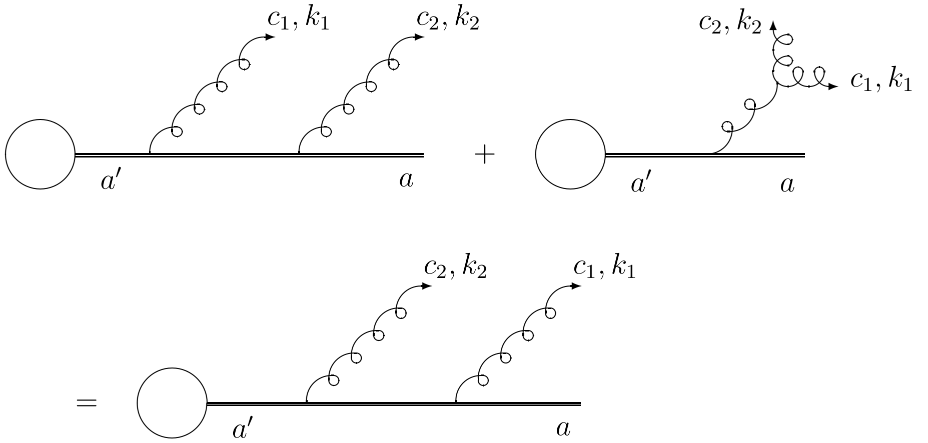

Fig. 3. Illustration of the soft gluon universality using the Jacobi identity

case of the more general strong angular ordering kinematics for which the soft factorization theorem was proven [14].

With this universal vertex combined with the Jacobi identity for the color matrices T^c the emission of the new soft particle from the "ends" a and c in Fig. 1 looks like if it were effectively emitted off the line a' at the left from the particle c. This is illustrated in Fig. 3.

Since the emitted gluon is real, $k^2 = \alpha\beta s - k_\perp^2 = 0$, and thus, combining (6) and (7), the vertex (8) can be rewritten in the form

$$\Gamma_c^\lambda(k) = gT^c \frac{2k_\perp \cdot \epsilon_\perp}{k_\perp^2}. \tag{9}$$

Repeating the new emissions from all "ends" in the same manner we come back to Fig. 2, where now the gluons are ordered according to their β value, the smaller β being on the left of the largest ones.

Note that the decreasing order of the β variables,

$$\beta_1 \gg \beta_2 \gg \cdots \gg \beta_n, \tag{10}$$

implies in the Regge kinematics an increasing order of the α,

$$\alpha_1 \ll \alpha_2 \ll \cdots \ll \alpha_n,$$

so that $\alpha_1 + \alpha_2 \simeq \alpha_2$, $\alpha_1 + \cdots + \alpha_n \simeq \alpha_n$. This property allows one to present the resulting soft cascade tree amplitude through an ordered exponent in momentum space,

$$\Phi_{\text{tree}} = P_\beta \exp\left\{\int_{x_\lambda}^1 \mathrm{d}\beta\, \mathrm{d}^2k_\perp V(\beta, k_\perp)\right\},$$

with

$$V(\beta, k_\perp) = V^+(\beta, k_\perp) + V^-(\beta, k_\perp) \tag{11}$$

and

$$V^\pm(\beta, k_\perp) = \frac{g}{(2\pi)^{\frac{3}{2}}} \frac{1}{2\beta} \frac{p_B \cdot \varepsilon^\lambda(\beta, k_\perp)}{p_B \cdot k} a^\pm_{\lambda,c}(\beta, k_\perp) T^c. \tag{12}$$

In the gauge $p_A \cdot A = 0$, this equation reduces to

$$V^\pm(\beta, k_\perp) = \frac{2gT^c}{(2\pi)^{\frac{3}{2}}} \frac{1}{2\beta} \frac{k_\perp \cdot \varepsilon_\perp^\lambda(\beta, k_\perp)}{k_\perp^2} a^\pm_{\lambda,c}(\beta, k_\perp), \tag{13}$$

where we have used (9). This expression incorporates both emission and absorption of the partons described by the light cone creation, $a^+_{\lambda c}(\beta, q_\perp)$, and annihilation, $a_{\lambda c}(\beta, q_\perp) \equiv a^-_{\lambda c}(\beta, q_\perp) = a^+_{\lambda c}(-\beta, -q_\perp)$, operators labelled by the longitudinal momentum fraction $\beta \geq 0$, transverse momentum $q_\perp$, and polarization and color indices λ and c, satisfying

$$\begin{aligned}&\left[a_{\lambda c}(\beta, q_\perp), a^+_{\lambda' c'}(\beta', q'_\perp)\right]\\ &= 2\beta\delta_{\lambda\lambda'}\delta_{cc'}\delta(\beta - \beta')\delta^{(2)}(q_\perp - q'_\perp).\end{aligned}$$

The ordering symbol P_β means that the fields $A(\beta)$ with the smallest β value are on the left from the fields with the largest ones. The minimal value $\beta = x_\lambda$ plays the role of an infrared cut-off in the amplitude Φ_{tree}. Note that in the case of an incoming line, the "time" variable z in the P-exponent (3) is Fourier conjugated to the longitudinal Sudakov variable α, and thus ordered oppositely. Since on the other hand, for mass-shell particles $\alpha \sim 1/\beta$, it turns out that $|z| \sim \beta$. Thus, the P-exponent defined with respect to z is in accordance with P_β.

When the cut-off value is lowered, that is when we take $x'_\lambda < x_\lambda$, it allows for the emission or absorption of an extra soft particle whose longitudinal momentum lies in the interval $x'_\lambda < \beta < x_\lambda$. This yields the new amplitude

$$\Phi'_{\text{tree}} = \left[1 + \int_{x_\lambda - \delta x_\lambda}^{x_\lambda} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V(\beta, k_\perp)\right] \Phi_{\text{tree}}, \tag{14}$$

the amplitude Φ_{tree} standing as the source for the new soft particles. The whole tree amplitude can be symbolically presented as an infinite product of elementary emissions or absorptions in the infinitesimal intervals Δx,

$$\Phi_{\text{tree}} = \prod_{x_\lambda}^{1} \left[1 + \int_{x-\Delta x}^{x} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V(\beta, k_\perp)\right]. \tag{15}$$

4 Virtual corrections and evolution of the cascade wavefunction

The previous form is convenient to include the virtual contributions. Besides the amplitude to emit or absorb a real gluon at each elementary step there is an amplitude for the case when the new phase volume in the longitudinal space remains non-occupied. In other words this means that the emitted gluon is reabsorbed at the same step, which corresponds to a loop correction. It does not change the number of particles, but it is responsible for the renormalization of the whole state.

Despite the composite internal structure of the whole system of cascade and source, within the soft emission approach this system looks like the source carrying the given momentum and color. The virtual correction results into self-energy insertion into the source propagator. In particular, if the source is gluon-like, then the full system looks like a gluon. In this case, the virtual correction results in a self-energy insertion in the gluon propagator. In LLA and for the cascade plus source system of momentum $q \simeq p_B + q_\perp$, the propagator of this system is modified as

$$D_{\mu\nu}(q) = \frac{1}{q^2}\left[\delta^{\mu\nu} - \frac{q^\mu p_A^\nu + q^\nu p_A^\mu}{p_A \cdot q}\right]\frac{1}{1-\pi(x_\lambda, q_\perp^2)},$$

with the function $\pi(x_\lambda, q_\perp^2)$ determining the z-factor, or normalization of the state. To keep the value of the norm fixed, the cascade plus source wavefunction is multiplied by $z^{-1/2}_{[x_\lambda,1]}(q_\perp)$, where $q_\perp$ is the total transverse momentum of the system whose constituents fill the interval of longitudinal momenta $[x_\lambda, 1]$,

$$z_{[x_\lambda,1]}(q_\perp) = \frac{1}{1-\pi(x_\lambda, q_\perp^2)}.$$

The contribution of the soft gluons appearing at the next evolution step in the interval $[x_\lambda - \delta x_\lambda, x_\lambda]$ should be compensated by the factor $z^{-1/2}_{[x_\lambda-\delta x_\lambda,x_\lambda]}(q_\perp)$. Basically for a finite interval one can write a loop expansion of the form

$$z_{[x_\lambda-\delta x_\lambda,x_\lambda]}(q_\perp) = 1 + \sum_{n>1} z_n(q_\perp)\ln^n \frac{x_\lambda}{x_\lambda - \delta x_\lambda},$$

only the first variation determining the evolution equation,

$$z^{-1/2}_{[x_\lambda-\delta x_\lambda,x_\lambda]}(q_\perp) = 1 - \omega(q_\perp)\frac{\delta x_\lambda}{x_\lambda} + O\left(\frac{\delta x_\lambda}{x_\lambda}\right)^2. \quad (16)$$

In LLA the function $\omega(q_\perp)$ is given by a one loop diagram calculated in the axial gauge.

Before discussing an explicit form of the virtual contribution note that the momentum $q_\perp$ in the argument of ω is the total transverse momentum of the system of cascade and source, which emit a soft particle as a whole. Introducing the total momentum operator $\hat{P}_\perp$, $\omega(\hat{P}_\perp)$ obviously gives $\omega(q_\perp)$ when acting on a whole system of cascade plus source state of total momentum $q_\perp$. Thus, in a similar way as in the tree case (see (14)), one step of evolution reads

$$\Phi(x_\lambda - \delta x_\lambda) = \left[1 - \omega(\hat{P}_\perp)\frac{\delta x_\lambda}{x} + \int_{x-\delta x_\lambda}^{x} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V(\beta, k_\perp)\right] \times \Phi(x_\lambda).$$

The full cascade S matrix can be symbolically written through the elementary steps product[1]

$$\Phi(x_\lambda) = \prod_{x_\lambda}^{1}\left[1 - \omega(\hat{P}_\perp)\frac{\Delta x}{x} + \int_{x-\Delta x}^{x} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V(\beta, k_\perp)\right], \quad (17)$$

where the brackets are ordered from the smallest β values at the left to the greatest ones at the right. The operator

$$\Phi^+(x_\lambda) = \prod_{x_\lambda}^{1}\left[1 - \omega(\hat{P}_\perp)\frac{\Delta x}{x} + \int_{x-\Delta x}^{x} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V^+(\beta, k_\perp)\right]$$

acting on the vacuum creates the soft constituents of the cascade, while the operator

$$\Phi^+(x_\lambda, p_\perp) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2z \mathrm{e}^{-\mathrm{i}p_\perp z}\mathrm{e}^{\mathrm{i}\hat{P}z}\Phi^+(x_\lambda)\mathrm{e}^{-\mathrm{i}\hat{P}z}$$

picks out the components with a fixed transverse momentum. Let us introduce the operator $b^+_{\sigma,a}(p_B, q_\perp)$ for creating the bare source, having transverse momentum $q_\perp$, and helicity and color indices σ and a, and denote this state by $|p_B, q_\perp, a, \sigma\rangle_s = b^+_{\sigma,a}(p_B, q_\perp)|0\rangle$. Then the state

$$|p_B, x_\lambda, q_\perp, a, \sigma\rangle = \int \mathrm{d}^2p_\perp \Phi^+(x_\lambda, p_\perp) b^+_{\sigma,a}(p_B, q_\perp - p_\perp)|0\rangle$$

describes the system of cascade plus source with given total transverse momentum $q_\perp$, cut-off value x_λ, and source helicity and color. The total tranverse momentum of the cascade is given by $p_\perp$. The operator $\hat{P}_\perp$ acting on the right results at each step in the transverse momentum of the total system at the previous cut-off value. The operator Φ is similar to the non-abelian coherent state operator which was elaborated in [15]. Indeed, as discussed above, our operator Φ is constructed through the β ordered product in the same manner as the non-abelian coherent state operator is constructed through an energy ordered product.

Changing x_λ we evidently have for the variation of the cascade plus source state

$$\delta|p_B, x_\lambda, q_\perp, a, \sigma\rangle = -\frac{\delta x_\lambda}{x_\lambda}\omega(q_\perp)|p_B, q_\perp, x_\lambda, a, \sigma\rangle + \int_{x_\lambda-\delta x_\lambda}^{x_\lambda} \mathrm{d}\beta \int \mathrm{d}^2k_\perp V^+(\beta, k_\perp)|p_B, (q-k)_\perp, x_\lambda, a, \sigma\rangle. \quad (18)$$

[1] Here δx_λ denotes the variation of the cut-off value while Δx is taken as the notation of an infinitisemal step in the infinite product representation of the cascade S matrix. Principally one can put $\delta x_\lambda = \Delta x$.

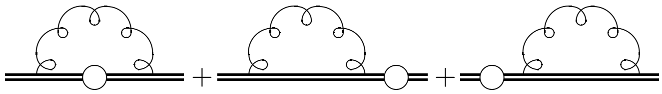

Fig. 4. Variation of the density with respect to the cut-off, in the case $x_Q \gg x_\lambda$. The double thick line stands for the system of source plus gluon cascade

Let Q be an operator, which probes the parton distribution in the cascade, for instance,[2]

$$Q = \int_{x_Q}^{1} \frac{\mathrm{d}x}{2x} \mathrm{d}^2 l_\perp f(x, l_\perp) a^+_{\sigma c}(x, l_\perp) a_{\sigma c}(x, l_\perp) \qquad (19)$$

with a weight function $f(x, l_\perp)$. The average value is generally expressed through the density function n_f, depending on the cascade total transverse momentum and cut-off,

$$\langle p_B, x_\lambda, q_\perp, a, \sigma | Q | p_B, x_\lambda, q'_\perp, a', \sigma' \rangle$$
$$= \delta_{a,a'} \delta_{\sigma,\sigma'} \delta^{(2)}(q_\perp - q'_\perp) n_f(x_\lambda, q_\perp).$$

Its evolution with x_λ value is determined by the variation of the state (18). If we suppose that the operator Q has a natural cut-off $x_Q \gg x_\lambda$, then the new emitted gluon does not affect the operator vertex (the operators $a_{\lambda,c}(\beta, k_\perp)$ commute with Q for $x_\lambda - \delta x_\lambda < \beta < x_\lambda$) and the variation of the matrix element decays into the sum

$$\varDelta n_f = \varDelta_1 n_f + \varDelta_2 n_f,$$

as illustrated in Fig. 4.

The first term is given by the extra particle amplitude squared times the average of the operator Q over the rest part of the amplitude corresponding to the previous cut-off value x_λ and recoil transverse momentum. It corresponds to the first diagram in Fig. 4 and can be written symbolically

$$\varDelta_1 n_f(x_\lambda, q_\perp) \qquad (20)$$
$$= \langle p_B, x_\lambda, q_\perp - k_\perp, a, \sigma | a_{\lambda,c} Q a^+_{\lambda,c} | p_B, x_\lambda, q'_\perp - k_\perp, a', \sigma' \rangle,$$

or in explicit form

$$\varDelta_1 n_f(x_\lambda, q_\perp)$$
$$= 2N_c \frac{g^2}{(2\pi)^3} \ln \frac{x_\lambda}{x'_\lambda} \int \frac{\mathrm{d}^2 k_\perp}{k^2_\perp} n_f(x_\lambda, q_\perp - k_\perp). \qquad (21)$$

Note that the previous equation is written in the case where the source is in the adjoint representation. In the general case, N_c should be replaced by the Casimir operator of the corresponding representation.

It is important to note that in soft loop calculations gluons can be considered as real, in accordance with the fact that there is almost no difference between real and virtual massless soft particles. Indeed, consider the loop where one single gluon is emitted and reabsorbed, as illustrated in Fig. 4. The α integral which occurs in the loops can be closed around the pole of the emitted gluon propagator. For, using the same technique as the one leading to the eikonal type expression (1), the denominator of the integrand reads $(\alpha - \mathrm{i}\delta)^2(\alpha\beta s - k^2_\perp + \mathrm{i}\delta)$ for the first graph of Fig. 4 or $(\alpha - \mathrm{i}\delta)(\alpha\beta s - k^2_\perp + \mathrm{i}\delta)$ for the second and third graphs. Both denominators leave the singularities in the α plane on the opposite side of the real axis. The numerator of the gluon propagator in the light cone gauge reads

$$d_{\mu\nu}(k) = -\varepsilon^\lambda_\mu(k) \varepsilon^\lambda_\nu(k) - \frac{4k^2}{\beta^2 s^2} p_{A\mu} p_{A\nu}.$$

When performing the α integral, if among the two poles one chooses to close around the physical pole of the gluon, the second term drops out since it cancels the α singularity in the propagator. Thus, only the first term remains. One then immediately gets the soft universal vertex squared (9) for the first graph.

The second term in the variation arises due to virtual corrections,

$$\varDelta_2 n_f(x_\lambda, q_\perp) = -2 \ln \frac{x_\lambda}{x'_\lambda} \omega(q_\perp). \qquad (22)$$

It is illustrated by the second and third diagrams of Fig. 4.

Differentiating with respect x_λ we arrive at the evolution equation

$$x_\lambda \frac{\partial}{\partial x_\lambda} n_f(x_\lambda, q_\perp)$$
$$= -2N_c \frac{g^2}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_\perp}{(q-k)^2_\perp} n_f(x_\lambda, k_\perp)$$
$$+ 2\omega(q_\perp) n_f(x_\lambda, q_\perp).$$

In the case where x_Q is smaller than the typical value of x_λ, there appears an additional term when the extra soft gluon operator is contracted with the operator Q, as shown by the additional fourth diagram in Fig. 5. It gives the following contribution to the variation of n_f:

$$\varDelta_3 n_f(x_\lambda, q_\perp) = 2N_c \frac{g^2}{(2\pi)^3} f(x_\lambda, q_\perp) \ln \frac{x_\lambda}{x'_\lambda}. \qquad (23)$$

The full inhomogeneous equation thus reads

$$x_\lambda \frac{\partial}{\partial x_\lambda} n_f(x_\lambda, q_\perp)$$
$$= 2N_c \frac{g^2}{(2\pi)^3} \int \frac{\mathrm{d}^2 l_\perp}{l^2_\perp} f(x_\lambda, l_\perp) \ln \frac{x_\lambda}{x'_\lambda}$$
$$- 2N_c \frac{g^2}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_\perp}{(q-k)^2_\perp} n_f(x_\lambda, k_\perp)$$

[2] In principle the operator Q could probe both the cascade content and the source, without modifying the following discussion. One could add to Q a term

$$1/2 \int \mathrm{d}^2 l_\perp f'(l_\perp) b^+_{\sigma,c}(p_B, l_\perp) b_{\sigma,c}(p_B, l_\perp)$$

acting on the source. If the source is gluon-like this is the same as to include in the weight function $f(x, l_\perp)$ in (19) a term proportional to $\delta(1-x)$.

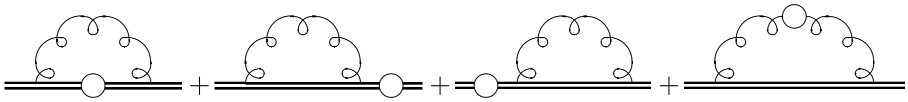

Fig. 5. Variation of the density with respect to the cut-off, in the case $x_Q \ll x_\lambda$

$$+ 2\omega(q_\perp) n_f(x_\lambda, q_\perp). \quad (24)$$

If one is interested in the density number of a gluon of momentum $r_\perp$ (integrating over longitudinal momentum fraction from the cut-off x_λ to 1), the weight function f in (19) should be chosen as

$$f(x, l_\perp) = \frac{1}{2}\delta^2(l_\perp - r_\perp), \quad (25)$$

the 1/2 factor being related to the average over gluon transverse polarization.

In that case, (24) turns into the equation

$$\begin{aligned} &x_\lambda \frac{\partial}{\partial x_\lambda} G(x_\lambda, r_\perp, q_\perp) \\ &= -2N_c \frac{g^2}{(2\pi)^3} \frac{1}{r_\perp^2} G_0(r_\perp, q_\perp) \\ &- 2N_c \frac{g^2}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_\perp}{(q-k)_\perp^2} G(x_\lambda, r_\perp, k_\perp) \\ &+ 2\omega(q_\perp) G(x_\lambda, r_\perp, q_\perp), \end{aligned} \quad (26)$$

where

$$G_0(r_\perp, q_\perp) = \frac{1}{2}\delta^2(q_\perp - r_\perp).$$

The inhomogeneous term, which is of lowest order, can be interpreted as the initial gluon contribution corresponding to the emission of one gluon off the bare source. It thus contains a factor $N_c g^2/(2\pi)^3$, multiplying $1/r_\perp^2$ which is reminiscent of the two t-channel gluon propagators (one being compensated by a polarization contribution after angular averaging of $k_\perp$).

Equation (26) looks like a Bethe–Salpeter equation with a kernel K acting from below, which is illustrated in Fig. 6.

Actually the density G is only a function of $p_\perp - r_\perp$ due to translational invariance. It means that (26) could equally be written as a Bethe–Salpeter equation with a kernel acting from above, by just an exchange between the variables $r_\perp$ and $p_\perp$.

A natural physical value for the IR cut-off x_λ is $x_\lambda \sim \mu^2/s$ where μ^2 is some typical scale for tranverse momentum. This follows from the Regge kinematics (4).

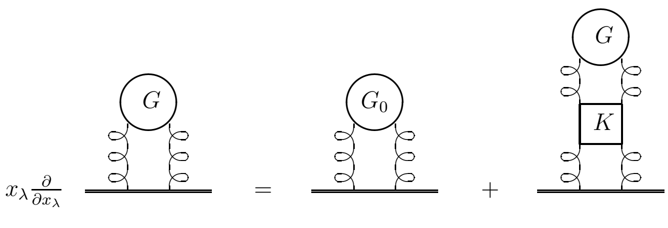

Fig. 6. Evolution equation for the integrated over x gluon density in a cascade. The double line stands for the source

Using (19) and (25), one can finally obtain the unintegrated parton density from G as

$$\begin{aligned} &g(x_\lambda, r_\perp, q_\perp) \\ &= \frac{1}{2x_\lambda} \langle p_B, x_\lambda, q_\perp, a, \sigma | \frac{1}{2} \sum_{\lambda, c} a^+_{\sigma,c} a_{\sigma,c} | p_B, x_\lambda, q'_\perp, a', \sigma' \rangle \\ &= -\frac{\partial}{\partial x_\lambda} G(x_\lambda, r_\perp, q_\perp). \end{aligned} \quad (27)$$

In the DGLAP case the unintegrated parton distributions, which depend both on the longitudinal and tranverse momenta, are related to the derivative of the structure functions with respect to the transverse cut-off. Similarly we have to consider the derivative with respect to x_λ of the function $G(x_\lambda, r_\perp, p_\perp)$ in order to get the unintegrated distribution. In (27), $r_\perp$ is the transverse momentum at which the parton density is measured while $q_\perp$ is the total transverse momentum of the cascade plus source. The function g satisfies the homogeneous equation

$$\begin{aligned} &x_\lambda \frac{\partial}{\partial x_\lambda} (x_\lambda g(x_\lambda, r_\perp, q_\perp)) \\ &= -2N_c \frac{g^2}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_\perp}{(q-k)_\perp^2} x_\lambda g(x_\lambda, r_\perp, k_\perp) \\ &+ 2\omega(q_\perp) x_\lambda g(x_\lambda, r_\perp, q_\perp). \end{aligned} \quad (28)$$

Both (26) and (28) have the same form as the BFKL equation for $t = 0$, although they describe gluon densities rather than high energy scattering amplitudes. Because of the universal nature of the soft vertex, these equations for the gluon densities can be written for any source (quark, gluon, ...). The only modification will be in the color factor N_c which should be replaced by the Casimir operator of the representation of $SU(N_c)$ to which the source belongs (C_F, C_A, ...). The same replacement should be made for the ω term, as will be shown in the next section.

5 Computation of ω

5.1 Evaluation of ω based on the gluon cascade universality in the case of a gluon-like source

The expression (21) provides a simple way to find the function $\omega(q_\perp)$ based on universality, in the peculiar case where the source is gluon-like. From the point of view of an emitted gluon, the whole system corresponding to the previous cut-off value plays the role of the source, and the z-factor is prescribed to this whole system. On the other hand, this whole system looks like a gluon (in the case where the bare source is gluon-like). Thus, the z-factor for the whole system and for a single gluon should have the same functional form. What we need is the virtual correction for the whole system, but we shall calculate instead the z-factor of the dressed emitted gluon. The inclusion of the z-factor corresponds to the replacement $a_{\lambda,\sigma} \to z^{1/2} a_{R\lambda,\sigma}$ in the expression (20). With the z-factor included (21) reads

$$\begin{aligned}
&\Delta_1 n_f(x_\lambda, q_\perp) \\
&= 2N_c \frac{g^2}{(2\pi)^3} \ln \frac{x_\lambda}{x_\lambda - \delta x_\lambda} \\
&\times \int \frac{d^2 p_\perp}{p_\perp^2} z_{[x_\lambda - \delta x_\lambda, x_\lambda]}(p_\perp)\, n_f(x_\lambda, q_\perp - p_\perp). \quad (29)
\end{aligned}$$

On the other hand there is a dual description of the same dressed gluon as a composite cascade state spread over the interval $[x_\lambda - \delta x_\lambda, x_\lambda]$ and carrying transverse momentum $p_\perp$. The two particle component of this state is given by the amplitude to emit two soft gluons off the source (which is the whole system corresponding to the previous cut-off value),

$$\begin{aligned}
&|p_B, q_\perp, x_\lambda - \delta x_\lambda, a\sigma\rangle_2 \\
&= \mathrm{i} g f_{ac_1d}\, \mathrm{i} g f_{dc_2b} \int\limits_{x_\lambda - \delta x_\lambda}^{x_\lambda} \frac{dx}{2x} \int\limits_{x_\lambda - \delta x_\lambda}^{x} \frac{d\beta}{2\beta} \int \frac{d^2 k_\perp}{(2\pi)^{3/2}} 2 \frac{k_\perp \cdot \varepsilon_{\sigma_2}}{k_\perp^2} \\
&\times \frac{1}{(2\pi)^{3/2}} 2 \frac{(p-k)_\perp \cdot \varepsilon_{\sigma_1}}{(p-k)_\perp^2} \\
&\times a^+_{\sigma_2 c_2}(\beta, k_\perp) a^+_{\sigma_1 c_1}(x, p_\perp - k_\perp) \\
&\times |p_B, q_\perp - p_\perp, x_\lambda, b, \sigma\rangle,
\end{aligned}$$

the gluons being ordered with respect to their longitudinal momenta,

$$x_\lambda - \delta x_\lambda < \beta < x < x_\lambda,$$

which is reflected in the integration limits.

Averaging the operator with this amplitude we arrive at the following expression for the gluon density variation:

$$\begin{aligned}
&\Delta_1 n_f(x_\lambda, q_\perp) \\
&= 2N_c \frac{g^2}{(2\pi)^3} \frac{1}{2} \ln^2 \frac{x_\lambda}{x_\lambda - \delta x_\lambda} \int \frac{d^2 p_\perp}{p_\perp^2} n_f(x_\lambda, q_\perp - p_\perp) \\
&\times 2N_c \frac{g^2}{(2\pi)^3} \int d^2 k_\perp \frac{p_\perp^2}{k_\perp^2 (p-k)_\perp^2}.
\end{aligned}$$

Note that the double integration in x and β results in $1/2 \ln^2 \frac{x_\lambda}{x_\lambda - \delta x_\lambda}$. This factor 1/2 reflects the Bose symmetry of the two gluons system. Recalling (16) and (29) we get

$$2\omega(p_\perp) = N_c \frac{g^2}{(2\pi)^3} \int d^2 k_\perp \frac{p_\perp^2}{k_\perp^2 (p-k)_\perp^2}. \quad (30)$$

5.2 Direct calculation of ω through the one loop self-energy of any source

The formula (30) can be compared with a direct calculation of the polarization operator. The exact one loop result for the gluon case in the dimensional regularization $(D = 2 + 2\epsilon)$ and axial gauge is

$$\begin{aligned}
\pi(p_\perp) &= -2N_c \frac{g^2}{8\pi^2} \Gamma\left(1 - \frac{D}{2}\right) \quad (31) \\
&\times \int_0^1 d\beta \left[\beta(1-\beta) p_\perp^2 / 4\pi\right]^{\frac{D}{2}-1} (1-\beta) K(\beta),
\end{aligned}$$

where

$$K(\beta) = \frac{\beta}{1-\beta} + \frac{1-\beta}{\beta} + \beta(1-\beta)$$

is the DGLAP kernel. The gluon self-energy is known to coincide in the axial gauge with a Sudakov form factor whose double logarithmic behavior originates from the product of transverse and longitudinal divergences. The latter one comes about when one gluon momentum in the loop becomes soft, $\beta \to 0$. To separate the longitudinal logarithms needed in order to find the function $\omega(p_\perp)$ we keep only the singular piece in $K(\beta)$ and cut the integral in (31) at $\beta = x_\lambda$, using dimensional regularization only for tranverse divergence, which turns into

$$\begin{aligned}
\pi_1(x_\lambda, p_\perp^2) &= -2N_c \frac{g^2}{8\pi^2} \Gamma\left(1 - \frac{D}{2}\right) \left[p_\perp^2 / 4\pi\right]^{\frac{D}{2}-1} \ln \frac{1}{x_\lambda} \\
&= 2\omega(p_\perp) \ln \frac{1}{x_\lambda}. \quad (32)
\end{aligned}$$

This results in

$$2\omega(p_\perp) = 2N_c \frac{g^2}{8\pi^2} \left[\frac{1}{\epsilon} + \ln \frac{p_\perp^2}{4\pi} + \gamma_E\right] + O(\epsilon),$$

while the dimensionally regularized integral (30) gives

$$\begin{aligned}
2\omega(q_\perp) &= N_c \frac{g^2}{8\pi^2} \left[q_\perp^2 / 4\pi\right]^{\frac{D}{2}-1} \Gamma\left(2 - \frac{D}{2}\right) \frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{\Gamma(D-2)} \\
&= 2N_c \frac{g^2}{8\pi^2} \left[\frac{1}{\epsilon} + \ln \frac{q_\perp^2}{4\pi} + \gamma_E\right] + O(\epsilon). \quad (33)
\end{aligned}$$

Thus, although the functions $\omega(p_\perp)$ defined by (30) and (32) look different their non-vanishing parts are equal.

Further, consider the renormalization of a heavy color source of mass M belonging to a given representation of $SU(N_c)$, irrespective of its spin representation. Due to classical current emission, the longitudinal divergent part (regularized by x_λ) of the source self-energy amounts to the same expression:

$$\pi_M(x_\lambda, p_\perp^2) = -2C \frac{g^2}{8\pi^2} \Gamma\left(1 - \frac{D}{2}\right) \left[\frac{M^2 - p^2}{4\pi}\right]^{\frac{n}{2}-1} \ln \frac{1}{x_\lambda}, \quad (34)$$

with source mass M and virtuality $M^2 - p^2 \simeq p_\perp^2$ regardless of the elementary or composite nature of the source. In the case of an arbitrary source, the color factor C is the appropriate Casimir operator of $SU(N_c)$.

The obtained expression reveals another type of "universality" of ω. Namely, except for the color factor, ω is identical for any kind of color source. It means that the reggeization of any source (electron [19,20], quark [21], gluon [1]) is universal. However, the whole equation for the scattering amplitude (not for the parton density) of arbitrary objects in the Regge limit generally differs from the BFKL equation (for example the quark–gluon scattering process with fermion exchange [21]).

6 QED cascade

The above treatment can be applied to the special case of electrodynamics. To avoid any confusion, we want to emphasize that the equation which will be obtained only describes the distribution of soft photons in a charged source. It has no direct relation to high energy scattering in the Regge limit.

Generally speaking, the wavefunction of the source, which could be any charged object, for instance an electron, is made of the bare source itself surrounded by a cloud of soft photons. The operator Q can probe either the source or photon component of the wavefunction. If we introduce a cut-off for the longitudinal momentum of the soft photons, we can study the effect of changing this cut-off as we did for the QCD case. We get the same three types of contributions as illustrated in Fig. 5, where the wavy lines now represent photons.

The z-factor is related in this case to the renormalization of the source, and can be obtained from (34). One needs only to replace g by the electric charge Ze and put $C = 1$. The real contribution is only due to the emission off the source line, since the photon has no charge. In the corresponding Wilson line (3), the ordering of the fields is not important since they commute at equal light cone time. It is related to the fact that in QED, the soft cascade structure does not assume any additional β ordering (10).

The equation satisfied by the corresponding n_f is the same as (24), after performing the replacement $g^2N_c \to Z^2e^2$. In the case of photon density, one gets

$$\begin{aligned} & x_\lambda \frac{\partial}{\partial x_\lambda} G_\gamma(x_\lambda, r_\perp, q_\perp) \\ & = -\frac{Z^2e^2}{(2\pi)^3}\frac{\delta^2(q_\perp - r_\perp)}{r_\perp^2} \\ & - 2\frac{Z^2e^2}{(2\pi)^3}\int \frac{\mathrm{d}^2k_\perp}{(q-k)_\perp^2} G_\gamma(x_\lambda, r_\perp, k_\perp) \\ & + 2\omega(q_\perp) G_\gamma(x_\lambda, r_\perp, q_\perp), \qquad (35) \end{aligned}$$

where $\omega(q_\perp)$ is obtained from the QCD case (see (30) and (33)) by the same replacement $g^2N_c \to Z^2e^2$. A similar equation for the unintegrated photon density g_γ could be written. It is obtained from (28) after performing the above modification.

These equations for the evolution of the densities have just the same form as the BFKL equation for QCD, although they have no direct relation with scattering since the latter is governed by the QED pomeron which is made of iterations of a kernel describing the elementary interaction process $2 \to 2$ t-channel photons through a quark box [22,23].

Let us stress once more that contrary to the QCD case, $\omega(q_\perp)$ corresponds to the reggeization of the source and not of the photon, which is of a different nature. In QCD, in the case where the source is gluon-like, $\omega(q_\perp)$ can be associated either to the source or to the cascade. We have used this property to get the expression for $\omega(q_\perp)$. From the point of view of source dressing, it is obtained through the one loop gluon self-energy (see (32)). Equivalently, it is obtained through the normalization of the wavefunction (see (30)).

7 Cascade wavefunction

Let us now turn back to the amplitude (17). Despite the symbolic character of the infinite ordered product it can be presented in a closed form. Consider to this end the amplitude to emit n quasi-real gluons with momenta $x_n, q_{\perp_n}, \ldots, x_1, q_{\perp_1}$, $x_1 > x_2 > \ldots > x_n$. It is given by the matrix element

$$\begin{aligned} & \Gamma_{\lambda_n,c_n,\ldots,\lambda_1,c_1;\sigma',\sigma;a',a}(x_n, q_{\perp n}, \ldots, x_1, q_{\perp 1}; q'_\perp, q_\perp) \\ & = \langle 0| b_{\sigma',a'}(p_B, q'_\perp) a_{\lambda_n,c_n}(x_n, q_{\perp_n}) \cdots a_{\lambda_1,c_1}(x_1, q_{\perp_1}) \\ & \quad \times \Phi^+(x_\lambda) b^+_{\sigma,a}(p_B, q_\perp)|0\rangle. \qquad (36) \end{aligned}$$

Expanding the brackets, the product in the $\Phi^+(x_\lambda)$ operator can be reorganized as

$$\begin{aligned} & \Phi^+(x_\lambda) \\ & = \prod_{x_\lambda}^{x_n}\left[1 - \omega(\hat{P})\frac{\Delta x}{x}\right]\int_{x_n-\Delta x}^{x_n} \mathrm{d}\beta_n \int \mathrm{d}^2k_{\perp_n} V^+(\beta_n, k_{\perp_n}) \\ & \times \prod_{x_{n-1}}^{x_n}\left[1 - \omega(\hat{P})\frac{\Delta x}{x}\right]\cdots\prod_{x_1}^{x_2}\left[1 - \omega(\hat{P})\frac{\Delta x}{x}\right] \\ & \times \int_{x_1-\Delta x}^{x_1} \mathrm{d}\beta_1 \int \mathrm{d}^2k_{\perp_1} V^+(\beta_1, k_{\perp_1})\prod_{x_1}^{1}\left[1 - \omega(\hat{P})\frac{\Delta x}{x}\right] \\ & + \cdots \end{aligned}$$

The terms which are not explicitly written give a zero contribution to the matrix element (36) when performing the contractions with a operators. Using the fact that

$$\prod_{x_2}^{x_1}\left[1 - \omega(\hat{P})\frac{\Delta x}{x}\right] = \exp\left\{-\int_{x_2}^{x_1}\frac{\mathrm{d}\beta}{\beta}\omega(\hat{P})\right\}$$

and that the operator $\hat{P}$ results in the total momentum of the state occurring to the right, we get for the amplitude

$$\begin{aligned} & \Gamma_{\lambda_n,c_n,\ldots,\lambda_1,c_1;\sigma',\sigma;a',a}(x_1, q_{\perp 1}, \ldots, x_n, q_{\perp n}; q'_\perp, q_\perp) \\ & = \left(\frac{x_\lambda}{x_n}\right)^{\omega(q_{\perp_n}+\cdots+q_{\perp_1}+q_\perp)} \Gamma^{\lambda_n}_{c_n}(q_{\perp n}) \\ & \times \left(\frac{x_n}{x_{n-1}}\right)^{\omega(q_{\perp_{n-1}}+\cdots+q_{\perp_1}+q_\perp)} \Gamma^{\lambda_{n-1}}_{c_{n-1}}(q_{\perp_{n-1}})\cdots \qquad (37) \\ & \times \Gamma^{\lambda_1}_{c_1}(q_{\perp_1})\left(\frac{x_1}{1}\right)^{\omega(q_\perp)} \end{aligned}$$

$$\times \langle 0|b_{\sigma',a'}(p_B, q'_\perp) b^+_{\sigma,a}(p_B, q_\perp)|0\rangle .$$

This formula, in the LLA approximation, provides all information on the cascade wavefunction. It resembles multi-reggeon formula but it includes only soft vertex. Note that the component of the cascade wavefunction without any real emission is nothing but the renormalization of the source wavefunction,

$$\sqrt{Z} = \left(\frac{1}{x_1}\right)^{-\omega(q_\perp)}, \tag{38}$$

as we discussed in Sects. 5 and 6.

8 Relationship with Sudakov form factor

The simplest application of the formula (37) for the cascade wavefunction is the calculation of the Sudakov form factor in the double logarithmic approximation. Let us consider the form factor of a source coupled to a colorless current carrying the momentum transfer q, in the kinematics where $Q^2 \equiv -q^2$ is very large with respect to the virtuality of the source. In this kinematics, it is convenient to choose the incoming particle momentum as being $p_B + p_\perp$ and the outgoing as being $p_A + p_\perp$, with $q^2 = -2p_A \cdot p_B$, as illustrated in Fig.7.

In the gauge $p_A \cdot A = 0$, this form factor only gets contributions from virtual loops along a p_B line. This is identical to the Z factor defined in (38). The Sudakov form factor S_F thus reads

$$S_F(p_\perp) = Z = \mathrm{e}^{-L}, \tag{39}$$

where

$$L = 2\omega(p_\perp) \ln \frac{1}{x_\lambda}. \tag{40}$$

This formula has only a symbolic meaning since $\omega(p_\perp)$ has IR logarithmic collinear divergences, which are not compensated by real emissions. However, Regge kinematics provides a natural cut-off for both collinear and soft divergences. Indeed, combining (30) or (33) (where $1/\epsilon$ corresponds to $\ln p_\perp^2$) with (40) and using the condition

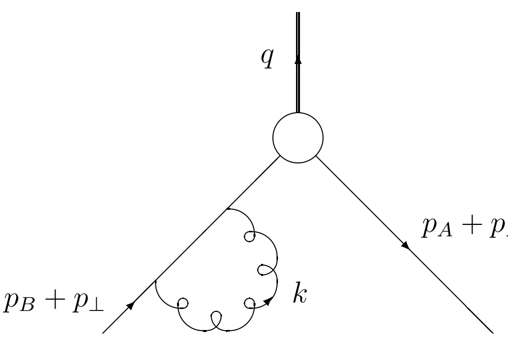

Fig. 7. Sudakov form factor for a source coupled to a virtual colorless object

$\beta s > k_\perp^2$ ($s = Q^2$), where $k \simeq \beta p_B + k_\perp$, L should be interpreted as

$$L = C_F \frac{\alpha_S}{\pi} \int_{x_\lambda}^{1} \frac{d\beta}{\beta} \int_{p_\perp^2}^{\beta Q^2} \frac{dk_\perp^2}{k_\perp^2}. \tag{41}$$

The infrared cut-off x_λ is defined by the Regge kinematics $\beta > p_\perp^2/s$. L thus reduces to

$$L = \frac{1}{2} C_F \frac{\alpha_S}{\pi} \ln^2 \frac{Q^2}{p_\perp^2}. \tag{42}$$

Since the source is quark-like, $C_A = N_c$ has been replaced by C_F in (30) or (33). One finally obtains

$$S_F = e^{-\frac{1}{2} C_F \frac{\alpha_S}{\pi} \ln^2 \frac{Q^2}{p_\perp^2}}. \tag{43}$$

In the QED case, $C_F = 1$ and α_S turns into $\alpha_{\mathrm{e.m.}}$.

Note that (43) is a simple case of the general double leading contribution arising from a cusp singularity, when the particle changes its direction from p_B to p_A [16,24].

9 Conclusion

This paper is based on the ordering of longitudinal variables in soft cascades, which means energy ordering of the emitted particles. It differs from the collinear approximation which implies angular ordering [25] and which is related to the conventional partonic DGLAP picture. We have shown explicitly that the infrared evolution equation for the parton density in the soft cascade is identical to the forward BFKL equation. The DGLAP equation is an evolution with respect to a transverse cut-off Q^2 due to collinear singularities, while in our case this evolution is written in terms of a longitudinal cut-off, due to soft singularities.

Our approach is based on a kind of duality in the description of a source at high energy. From the viewpoint of the soft vertex the source can be treated as a single particle with given momentum and color. On the other hand it has an internal structure, a cloud of many soft gluons surrounding it. It looks like a composite object with a non-trivial wavefunction. Thus a rather simple interpretation of reggeization as a manifestation of a soft cascade structure appears in this picture. The trajectory $\omega(q_\perp)$ is nothing more than the source Z factor in the axial gauge. Both in QCD and QED, this function is associated with the charged source self-energy. It is universal in the sense that it is the same for all sources, up to a global color factor. In the DGLAP case the divergences occurring at $x \to 1$ are regulated by the $1/(x-1)_+$ prescription, corresponding to UV Z-factors due to UV renormalization of parton wavefunctions. In (26) and (28) the IR divergences for $k_\perp \to 0$ disappear because of virtual corrections which can be treated as an IR renormalization of the source wavefunction.

The equations that we have obtained describe the evolution of the gluon (or photon) density. The extension of this approach for scattering amplitudes, for both the forward and non-forward case, as well as for the generalized leading log approximation [26–28], will be presented in a forthcoming paper. It would also be interesting to study more complicated structures like for example the triple pomeron vertex.

Finally, let us emphasize that our treatment relies on the resummation of s-channel soft gluons (or photons). It is very similar to the dipole model, except that it does not assume any large N_c limit. On the other hand, this s-channel picture is very similar to the $x_{\mathrm{Bj}} \to 1$ limit in DIS. This relation will be explored elsewhere.

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